MMAT5360 200406

n-person cooperative games

$\mathcal{A} = \{1, 2, \cdots, n\}$ set of players

$S \subset \mathcal{A}$: coalition

$S^c = \mathcal{A} \setminus S$: counter coalition of S.

Characteric function : $v : \mathcal{P}(\mathcal{A}) \longrightarrow \mathbb{R}$ s.t.

1. $v(\emptyset) = 0$

2. Superaddivity : If $S \cap T \neq \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$

Imputation : $\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$

1. $x_i \geq v(\{i\})$ for $i = 1, 2, \cdots, n$

2. $x_1 + x_2 + \cdots + x_n = v(\mathcal{A})$

$(\mathcal{A}, v)$ is essential if $\sum_i v(\{i\}) < v(\mathcal{A})$

$(\mathcal{N}, v)$ is essential if $\sum_{i \in \mathcal{N}} v(\{i\}) < v(\mathcal{N})$

is inessential if $\sum_{i \in \mathcal{N}} v(\{i\}) = v(\mathcal{N})$

$\vec{x} \in C(v)$ if for any $S \subset \mathcal{N}$, $\sum_{i \in S} x_i \geq v(S)$

Example: Suppose $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

$\mathcal{N} = \{1, 2, 3\}$

S	v
$\{1,2\}$	$\frac{1}{3}$
$\{1,3\}$	$\frac{1}{2}$
$\{2,3\}$	$\frac{1}{4}$
$\{1,2,3\}$	1

$v(\phi) = 0$

$$\vec{x} \in C(v) \iff \begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 \geq \frac{1}{3} \\ x_1 + x_3 \geq \frac{1}{2} \\ x_2 + x_3 \geq \frac{1}{4} \end{cases} \implies \begin{cases} x_1 + x_2 + x_3 = 1 \\ 0 \leq x_1 = 1 - (x_2 + x_3) \leq \frac{3}{4} \\ 0 \leq x_2 = 1 - (x_1 + x_3) \leq \frac{1}{2} \\ 0 \leq x_3 = 1 - (x_1 + x_2) \leq \frac{2}{3} \end{cases}$$

$x_1, x_2, x_3 \geq 0$

Represent $C(v)$ on the x_1-x_2 plane

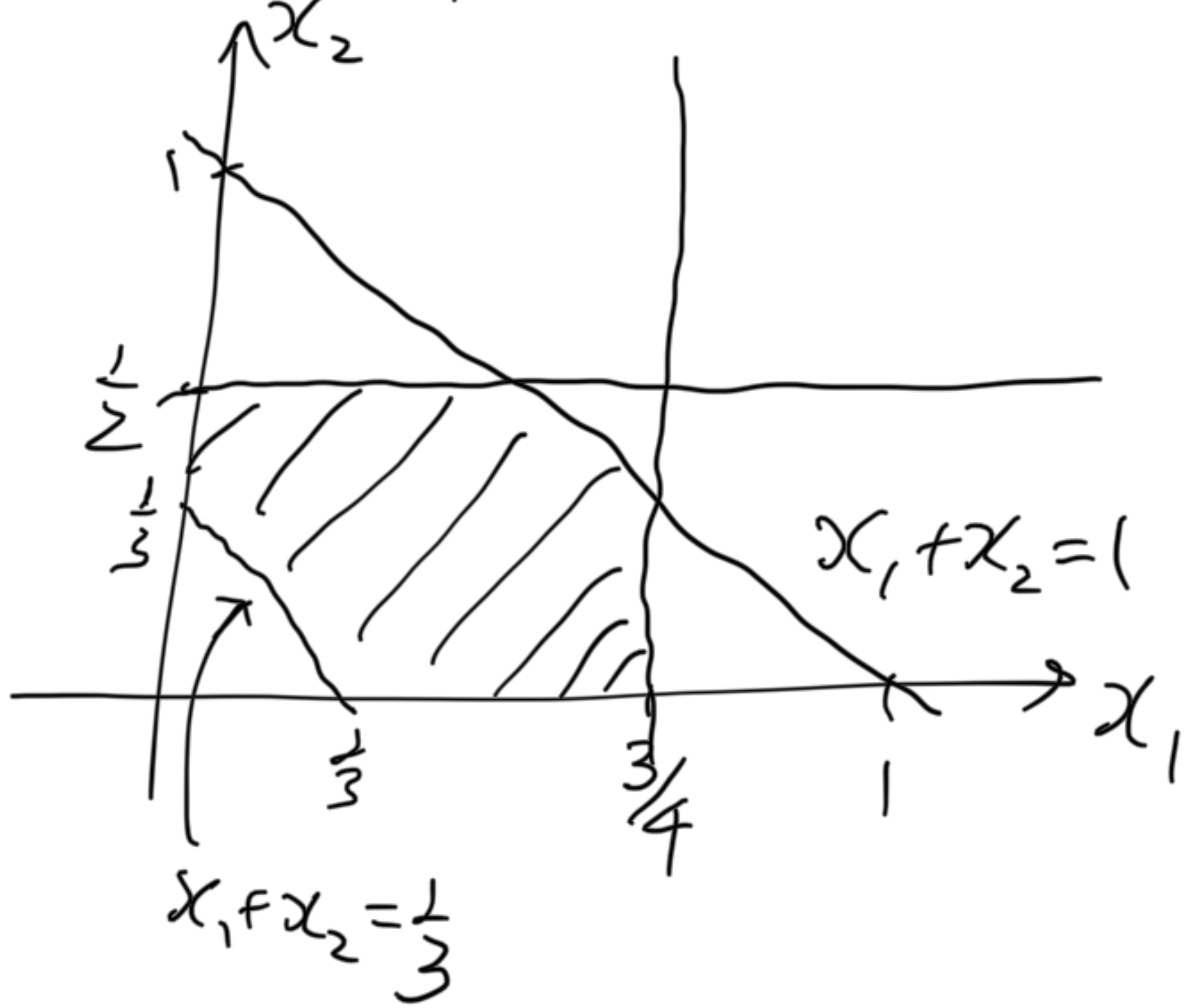
$x_3 = 1 - (x_1 + x_2)$

$0 \leq x_1 \leq \frac{3}{4}$

$0 \leq x_2 \leq \frac{1}{2}$

$x_1 + x_2 \geq \frac{1}{3}$

$x_1 + x_2 = 1 - x_3 \leq 1$



Example (Use car game) $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

S	$v(S)$
$\{1,2\}$	500
$\{1,3\}$	700
$\{2,3\}$	0

$\{1,2,3\}$ $\quad 700$

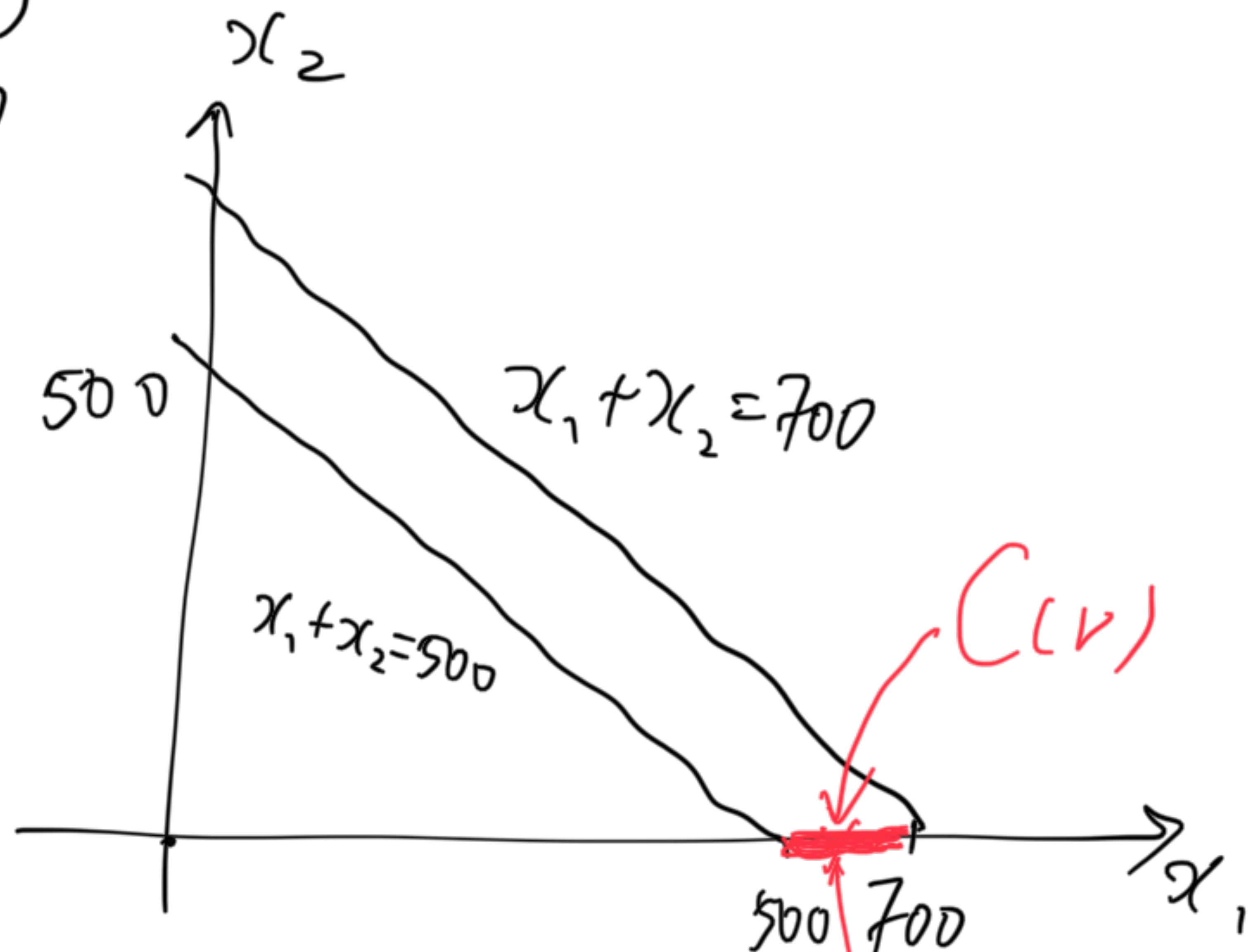
$$\vec{x} \in C(v) \iff \begin{cases} x_1 + x_2 + x_3 = 700 \\ x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 500 \\ x_1 + x_3 \geq 700 \\ x_2 + x_3 \geq 0 \end{cases}$$

$x_1 - x_2$ plane

$$0 \leq x_1 = 700 - (x_2 + x_3) \leq 700$$

$$0 \leq x_2 = 700 - (x_1 + x_3) \leq 0$$

$$500 \leq x_1 + x_2 = 700 - x_3 \leq 700$$



Example: 3-person constant sum game

S	$v(S)$
$\{1\}$	$\frac{1}{4}$
$\{2\}$	$-\frac{1}{3}$
$\{3\}$	0

$$x_1 + x_2 + x_3 = 1$$

$$\frac{1}{4} \leq x_1 \leq 1 - (x_2 + x_3) = \frac{1}{4} \Rightarrow x_1 = \frac{1}{4}$$

$\{3\}$	0	
$\{1,2\}$	1	$-\frac{1}{3} \le x_2 \le 1-(x_1+x_3) = -\frac{1}{3} \Rightarrow x_2 = -\frac{1}{3}$
$\{1,3\}$	$\frac{4}{3}$	$1 \le x_1 + x_2 = 1 - x_3 \le 1$
$\{2,3\}$	$\frac{3}{4}$	No solution
$\{1,2,3\}$	1	$C(\nu) = \phi$

(0,1)-reduced form

Def. Two characteristic functions ν and μ are strategically equivalent if $\exists\, k>0$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$ s.t.

$$\mu(S) = k\nu(S) + \sum_{i\in S} c_i$$

Thm. Suppose ν and μ are equivalent.

1. μ is essential $\Leftrightarrow$ ν is essential
2. $I(\mu) = \{\vec{y} : \vec{y} = k\vec{x} + \vec{c},\ \vec{x} \in I(\nu)\}$

2. $I(\mu) = \{\vec{y} : \vec{y} = k\vec{x} + \vec{c},\ \vec{x} \in I(\nu)\}$

$\vec{c} = (c_1, c_2, \cdots, c_n)$

3. $C(\mu) = \{\vec{y} : \vec{y} = k\vec{x} + \vec{c},\ \vec{x} \in C(\nu)\}$

Def. We say μ is a $(0,1)$-reduced form if

1. $v(\{i\}) = 0$ for any $i = 1, 2, \cdots, n$
2. $v(N) = 1$

Thm. 1. Every inessential game is equivalent to the zero game $(v(S) = 0)$.

2. Every essential game is equivalent to a unique game in $(0,1)$ reduced form.

Proof. 1. (omitted)

2. Define $\mu(S) = \dfrac{v(S) - \sum_{i \in S} v(\{i\})}{\quad}$

$$v(\mathcal{A}) - \sum_{i\in\mathcal{A}} v(\{i\})$$

Note v is essential game $\Rightarrow v(\mathcal{A}) - \sum_{i\in\mathcal{A}} v(\{i\}) > 0$

Easy to check μ is a $(0,1)$-reduced form

$$\left(k = \frac{1}{v(\mathcal{A}) - \sum_{i\in\mathcal{A}} v(\{i\})} > 0 \qquad c_i = -\frac{v(\{i\})}{v(\mathcal{A}) - \sum_{i\in\mathcal{A}} v(\{i\})} \right)$$

Example: 3-person constant sum game

$$\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 0, \quad \mu(\mathcal{A}) = 1$$

$$\mu(\{1,2\}) = \frac{v(\{1,2\}) - (v(\{1\}) + v(\{2\}))}{v(\mathcal{A}) - (v(\{1\}) + v(\{2\}) + v(\{3\}))}$$

$$= \frac{1 - (\frac{1}{4} + (-\frac{1}{3}))}{1 - (\frac{1}{4} + (-\frac{1}{3}) + 0)} = 1$$

$$\mu(\{1,3\}) = \frac{\cdots}{1-\left(\frac{1}{4}+\left(-\frac{1}{3}\right)+0\right)} = 1$$

$$\mu(\{2,3\}) = 1$$

Example (Used car game)

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$$

$$v(\{1,2\}) = 500,\quad v(\{1,3\}) = 700,\quad v(\{2,3\}) = 0$$

$$v(\{1,2,3\}) = 700$$

$$\mu(\{1,2\}) = \frac{v(\{1,2\}) - (v(\{1\}) + v(\{2\}))}{v(N) - (v(\{1\}) + v(\{2\}) + v(\{3\}))}$$

$$= \frac{500-0}{700-0} = \frac{5}{7}$$

$$\mu(\{1,3\}) = \frac{700-0}{700} = 1$$

$\mu(\{2,3\}) = 0$

$\mu(A) = 1$

Shapley values

Def. Shapley value of player i is

$$\phi_i = \sum_{S \in \mathcal{P}(A)\setminus\{\emptyset\}} \frac{(n-|S|)!\,(|S|-1)!}{n!}\,\delta(i,S)$$

where $\delta(i,S) = v(S) - v(S\setminus\{i\})$

$\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)$ Shapley vector

Remark:

1. $\delta(i,S) = v(S) - v(S\setminus\{i\})$ is the amount

i contributes to the coalition S

i contributes to the coalition S.

If $i \notin S$, $\delta(i, S) = 0$.

To find ϕ_i, only need to sum over $i \in S$

2. $$\phi_i = \sum_{S \in \mathcal{P}(A)} \frac{(n-|S|-1)!\,|S|!}{n!} \left(v(S \cup \{i\}) - v(S) \right)$$

3. ϕ_i measures the <u>average</u> contribution over the possible permutations in which the grand coalition can be form.

$n=3$ Permutation	$S \setminus \{1\}$	S	$\delta(1, S)$
1 2 3	$\emptyset$	$\{1\}$	$v(\{1\})$
1 3 2	$\emptyset$	$\{1\}$	$v(\{1\})$
2 1 3	$\{2\}$	$\{1,2\}$	$v(\{1,2\}) - v(\{2\})$
2 3 1	$\{2,3\}$	A	$v(A) - v(\{2,3\})$

3 1 2 $\{3\}$ $\{1,3\}$ $v(\{1,3\}) - v(\{3\})$

3 2 1 $\{2,3\}$ $\mathcal{A}$ $v(\mathcal{A}) - v(\{2,3\})$

$$\phi_1 = \frac{1}{3!}\Big(2 \times v(\{1\}) + \big(v(\{1,2\}) - v(\{2\})\big) + \big(v(\{1,3\}) - v(\{3\})\big)$$
$$+ 2\big(v(\mathcal{A}) - v(\{2,3\})\big)\Big)$$

For general n, fixed $S \subset \mathcal{A}$

What is the weight of $\delta(i, S)$?

For $\sigma \in S_n$ permutation $s = |S|$

$$\underbrace{\sigma(1), \sigma(2), \cdots, \sigma(s-1)}_{\text{in } S \setminus \{i\}},\ i,\ \underbrace{\sigma(s+1), \cdots, \sigma(n)}_{S^c = \mathcal{A} \setminus S}$$

$$(s-1)! \quad \times \quad (n-s)!$$

Example: 3-person constant sum game

Consider coalitions containing 1.

$\{1\}$, $\{1,2\}$, $\{1,3\}$, $A=\{1,2,3\}$

$$\phi_1 = \frac{(3-1)!(1-1)!}{3!}(v(\{1\})-v(\phi)) + \frac{(3-2)!(2-1)!}{3!}(v(\{1,2\})-v(\{2\}))$$

$$+ \frac{(3-2)!(2-1)!}{3!}(v(\{1,3\})-v(\{3\})) + \frac{(3-3)!(3-1)!}{3!}(v(A)-v(\{2,3\}))$$

$$= \frac{11}{18}$$

Similarly $\phi_2 = \frac{1}{36}$. $\phi_3 = \frac{13}{36}$

Note. $\phi_1+\phi_2+\phi_3 = 1 = v(A)$

$n=7$ $S=\{1,3,4,5\}$. $S^c=\{2,6,7\}$

$$\delta(1, S) = v(S) - v(\{3,4,5\})$$

$$7! = 5040$$

permutation of 3,4,5   permutation of 2,6,7

$$(4-1)! = 3! = 6 \left\{ \begin{matrix} 3\ 4\ 5 \\ 3\ 5\ 4 \\ 4\ 3\ 5 \end{matrix} \right. \quad , 1 , \quad \left. \begin{matrix} 2,6,7 \\ 6,2,7 \\ \vdots \end{matrix} \right\} (7-4)! = 3! = 6$$

$6 \times 6 = 36$ terms